Journal of Applied Mechanics and Technical Physics, Vol. 36, No. 2, 1995

STEADY OSCILLATIONS OF A PLANE PROFILE NEAR THE INTERFACE OF MEDIA WITH A FLAT BOTTOM

S. I. Filippov

UDC 532.5

We study in the framework of small-amplitude wave theory the steady oscillations of a profile in a two-layer ideal incompressible ponderable fluid flow that is bounded by a flat bottom. Even in the linear formulation, the solution of this problem causes great difficulties, associated in considerable measure with satisfaction of the boundary condition on the profile contour. Among the methods for solving the problems of the oscillations of bodies in a fluid with media interfaces, we note the Kochin method [1], based on the distribution of the singularities around the contour. Khaskind [2], using the approach of [1], examined the problem of the oscillations of a profile beneath the free surface of a liquid of finite depth. Approximate expressions were obtained in [2] for the forces acting on the profile with use of the Kochin hypothesis. The numerical methods have recently been widely used to solve the oscillation problems. The motions of bodies in flows with a single interface were studied in [3, 4] using a hybrid finite-element method.

In the present work we develop the numerical-analytical method for modeling boundaries with singularities [5-7], proposed by Tumashev. One of the primary advantages of the method lies in the satisfaction of the boundary condition on the profile in the course of the construction of the solution. This method was used previously in [7] for flows with two interfaces in studying the translational motion of a wing profile.

1. Statement of the Problem. We examine the wave motions that arise in a ponderable fluid in the case of periodic oscillations of a plane profile beneath the line of separation of fluids of different density with the presence of a flat bottom. We shall assume that the waves that form on the line of separation propagate in both directions from the body.

Let xOy be the stationary coordinate system. The x axis is horizontal and coincides with the undisturbed level of the line of separation, and the y axis is directed upward. By virtue of the linearity of the problem, it is sufficient to examine harmonic oscillations of the profile C satisfying the law

$$u_n(s,t) = u_1(s)\cos\omega t + u_2(s)\sin\omega t.$$

Here $u_n(s, t)$ is the normal component of the velocity of the profile point with the arcwise abscissa s; t is the time.

Considering the oscillations to be steady, we can write the complex velocity potential in the form

$$w(z,t) = w_1(z)\cos\omega t + w_2(z)\sin\omega t \quad (z = x + iy).$$

Since the amplitudes of the oscillations are small, the flow tangency condition

$$\frac{\partial\varphi_k}{\partial n} = u_k(s) \qquad (w_k(z) = \varphi_k(x,y) + i\psi_k(x,y)) \quad (k = 1, 2) \tag{1.1}$$

applies to the contour C, assumed to be stationary.

The boundary conditions on the line of separation and the bottom line for the functions $w_k(z)$ take the form

$$\operatorname{Im}[dw_k^+(z)/dz + i(m^+ w_k^+(z) - m^- w_k^-(z))] = 0, \ \ y = 0; \tag{1.2}$$

$$\operatorname{Im}[w_k^+(z) - w_k^-(z)] = 0, \ \ y = 0; \tag{1.3}$$

$$\operatorname{Im} w_k(z) = \psi_0, \ \ y = -H \quad (\psi_0 = \text{const}), \tag{1.4}$$

Scientific Research Institute of Mathematics and Mechanics of Kazan State University, 420008 Kazan. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 2, pp. 39-44, March-April, 1995. Original article submitted March 30, 1994; revision submitted May 6, 1994.

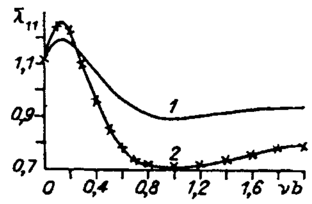

Fig. 1

where $m^+ = \rho^+\omega^2/g(\rho^+ - \rho^-)$; $m^- = \rho^-\omega^2/g(\rho^+ - \rho^-)$; g is the gravity force acceleration; the plus symbol denotes the quantities relating to the fluid surrounding the body, the minus symbol relates to the region $y \geq 0$.

2. Derivation of the Integral Equations. It is easy to establish that the function $\overset{+}{\varphi}_k(x, y)$ is single-valued in the region outside the contour E(C) [2], consequently $\overset{+}{w}_k(z)$ when circling the contour C can change only by the imaginary cyclic constant iA_k, where $A_k = \oint\limits_C u_k(s)ds$. We shall first assume that $A_k = 0$.

We shall seek the solution of the posed problem in the form

$$w_k = w_k^0(z) + w_k^*(z) + v_k(z) + \tilde{v}_k(z) + \Phi_k(z) + \tilde{\Phi}_k(z). \tag{2.1}$$

Here $\overset{0}{w}_k(z)$ are known functions, representing the solution of the problem for an unbounded flow; $\overset{*}{w}_k(z) = 0$ for $A_k = 0$;

$$v_k(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\mu_k(\tau)\,d\tau}{z-\tau}; \qquad \tilde{v}_k(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\tilde{\mu}_k(\tau_1)\,d\tau_1}{z-\tau_1+iH};$$

$$\Phi_k(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}F(z,\tau)\mu_k(\tau)\,d\tau; \quad \tilde{\Phi}_k(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\tilde{F}(z,\tau_1)\tilde{\mu}_k(\tau_1)\,d\tau_1.$$

By $\tilde{\mu}_k(\tau_1)$ we shall mean $\tilde{\mu}_k(\tau_1 - iH)$. We shall consider that the real functions $\mu_k(\tau)$, $\tilde{\mu}_k(\tau_1)$ are such that the integrals $v_k(z)$ and $\tilde{v}_k(z)$ converge. The functions $F(z, \tau)$, $\tilde{F}(z, \tau_1)$ are constructed on the basis of the Milne-Thompson circle theorem with the use of conformal mapping of the exterior of the contour onto the exterior of a circle [6, 7] so that $w_k(z)$ in the form (2.1) satisfies the condition (1.1). Moreover, on the basis of the properties of the limiting values of the Cauchy-type integral, the condition (1.3) is satisfied.

Substituting (2.1) into the condition (1.2), we obtain

$$\mathrm{Im}\left\{\frac{dv_k}{dz} + i\nu v_k + \left(\frac{d}{dz} + i\sigma\right)[\Omega_k(z)]\right\} = 0, \quad y = 0,$$

where $\Omega_k(z) = \overset{0}{w}_k(z) + \tilde{v}_k(z) + \Phi_k(z) + \tilde{\Phi}_k(z)$; $\sigma = m^+ - m^-$; $\nu = m^+ + m^-$. This relation is equivalent to the following:

$$\mathrm{Im}\left\{\frac{dv_k}{dz} + i\nu v_k - \left(\frac{d}{dz} - i\sigma\right)[\overline{\Omega}_k(z)]\right\} = 0, \quad y = 0.$$

Since the functions $v_k(z)$, $\overset{0}{\overline{w}}_k(z)$, $\bar{\tilde{v}}_k(z)$, $\overline{\Phi}_k(z)$, $\overline{\tilde{\Phi}}_k(z)$ are regular in the lower halfplane, then we have for the given region

$$\frac{dv_k}{dz} + i\nu v_k - \left(\frac{d}{dz} - i\sigma\right)[\overline{\Omega}_k(z)] = B_k \tag{2.2}$$

(B_k are real constants). Without loss of generality, we set $B_k = 0$. Consequently, from (5) we can find

$$v_k(z) = -m_1\overline{\Omega}_k(z) + e^{-i\nu z}\left[\frac{C_k}{2} + 2m_2\int\limits_{\infty}^{z} e^{i\nu z}\frac{d\overline{\Omega}_k(u)}{du}\,du\right]. \tag{2.3}$$

Here C_k are unknown constants; $m_1 = (\rho^+ - \rho^-)/(\rho^+ + \rho^-)$; $m_2 = \rho^+ \text{-}/(\rho^+ + \rho^-)$.

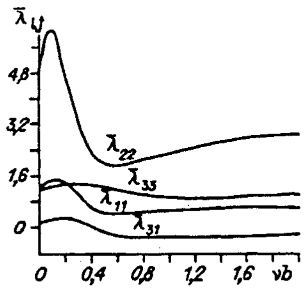

Fig. 2

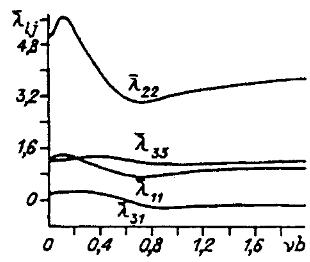

Fig. 3

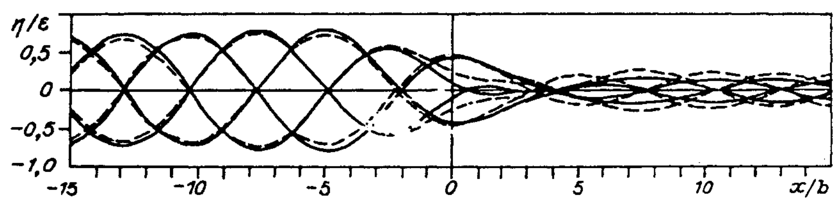

Fig. 4

Performing in (2.3) the limit passage with $z = z_1 - iH$ on the basis of the Sokhotskii formula and examining the real part of the equality, we obtain the linear integral equation

$$
\begin{aligned}
&\mu_k(x) = \mathrm{Re}\,(C_k e^{-i\nu x}) + q_k(x) + \\
&\qquad + \int_{-\infty}^{\infty} K_1(x,\tau)\mu_k(\tau)\,d\tau + \int_{-\infty}^{\infty} K_4(x,\tau_1)\tilde{\mu}_k(\tau_1)\,d\tau_1, \\
&q_k(x) = 2\mathrm{Re}\Big[-m_1 w_k^0(x) + 2m_2 e^{i\nu x}\int_{\infty}^{x} \frac{dw_k^0(u)}{du} e^{-i\nu u}\,du\Big], \\
&K_1(x,\tau) = \frac{1}{\pi}\,\mathrm{Im}\Big[-m_1 F(x,\tau) + 2m_2 e^{i\nu x}\int_{\infty}^{x} F'_u(u,\tau)e^{-i\nu u}\,du\Big], \\
&K_4(x,\tau_1) = \frac{1}{\pi}\,\mathrm{Im}\Big\{\frac{-m_1}{x-\tau_1+iH} - m_1\tilde{F}(x,\tau_1) + \\
&\qquad + 2m_2 e^{i\nu x}\int_{\infty}^{x}\Big[\frac{-1}{(u-\tau_1+iH)^2} + \tilde{F}'_u(u,\tau_1)\Big]e^{-i\nu u}\,du\Big\}.
\end{aligned}
\tag{2.4}
$$

Performing the coordinate transformation $z = z_1 - iH$ and substituting (2.1) into the condition (1.4), we find

$$
\begin{aligned}
&\tilde{\mu}_k(x_1) = \tilde{q}_k(x_1) + \int_{-\infty}^{\infty} K_2(x_1,\tau_1)\tilde{\mu}_k(\tau_1)\,d\tau_1 + \int_{-\infty}^{\infty} K_3(x_1,\tau)\mu_k(\tau)\,d\tau, \\
&\tilde{q}_k(x_1) = -2\mathrm{Re}\,[w_k^0(x_1)], \quad K_2(x_1,\tau_1) = \frac{1}{\pi}\,\mathrm{Im}\tilde{F}(x_1,\tau_1), \\
&K_3(x_1,\tau) = \frac{1}{\pi}\,\mathrm{Im}\Big[\frac{1}{x_1-\tau-iH} + F(x_1,\tau)\Big].
\end{aligned}
\tag{2.5}
$$

We introduce the notations

$$
H_k(\nu) = \int_{-\infty}^{\infty} \frac{d}{dx}[w_k^0(x) + w_k^*(x) + \tilde{v}_k(x) + \Phi_k(x) + \tilde{\Phi}_k(x)]e^{-i\nu x}\,dx,
$$

then it is not difficult to obtain from the radiation condition similarly to [6]

$$C_1 = -2m_2[\overline{H}_1(\nu) - i\overline{H}_2(\nu)], \quad C_2 = iC_1. \tag{2.6}$$

We shall examine the systems

$$\begin{aligned} \mu_{k0}(x) &= q_k(x) + \int_{-\infty}^{\infty} K_1(x,\tau)\mu_{k0}(\tau)\,d\tau + \int_{-\infty}^{\infty} K_4(x,\tau_1)\tilde{\mu}_{k0}(\tau_1)\,d\tau_1, \\ \tilde{\mu}_{k0}(x_1) &= \tilde{q}_k(x_1) + \int_{-\infty}^{\infty} K_2(x_1,\tau_1)\tilde{\mu}_{k0}(\tau_1)\,d\tau_1 + \int_{-\infty}^{\infty} K_3(x_1,\tau)\mu_{k0}(\tau)\,d\tau, \end{aligned} \tag{2.7}$$

where k = 0, 1, 2; $q_0(x) = \exp(i\nu x)$; $\tilde{q}_0(x_1) = 0$.

We can write the solution of the system (2.4), (2.5) in terms of the solutions of the systems (2.7):

$$\begin{aligned} \mu_k(x) &= \mu_{k0}(x) + \operatorname{Re}[C_k\bar{\mu}_{00}(x)], \\ \tilde{\mu}_k(x_1) &= \tilde{\mu}_{k0}(x_1) + \operatorname{Re}[C_k\bar{\tilde{\mu}}_{00}(x_1)]. \end{aligned} \tag{2.8}$$

Considering the relations (2.8) and the fact that $C_2 = iC_1$, we find

$$\mu_1(x) + i\mu_2(x) = \mu_{10}(x) + i\mu_{20}(x) + C_1\mu_{00}(x),$$

then

$$H_1(\nu) + iH_2(\nu) = H_{10}(\nu) + iH_{20}(\nu) + \overline{C}_1(\Delta\mu_{00} + \Delta\tilde{\mu}_{00}). \tag{2.9}$$

Here

$$\begin{aligned} H_{k0}(\nu) &= \int_{-\infty}^{\infty} \frac{d}{dx}[w_k^0(x) + \overset{*}{w}_k(x)]e^{-i\nu x}\,dx + \Delta\mu_{k0}(\nu) + \Delta\tilde{\mu}_{k0}(\nu); \\ \Delta\mu_{k0}(\nu) &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-i\nu x}\int_{-\infty}^{\infty} F_x'(x,\tau)\mu_{k0}(\tau)\,d\tau dx; \\ \Delta\tilde{\mu}_{k0}(\nu) &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-i\nu x}\int_{-\infty}^{\infty} \Big[\frac{-1}{(x-\tau_1+iH)^2} + \tilde{F}_x'(x,\tau_1)\Big]\tilde{\mu}_{k0}(\tau_1)\,d\tau_1 dx. \end{aligned}$$

Substituting the expression that is conjugate to (2.9) into (2.6), we obtain

$$C_1 = \frac{-2m_2[\overline{H}_{10}(\nu) - i\overline{H}_{20}(\nu)]}{1 + 2m_2(\overline{\Delta\mu}_{00} + \overline{\Delta\tilde{\mu}}_{00})}.$$

In the case $A_k \neq 0$ we represent $\overset{*}{w}_k(z)$ by the multivalued in E(C) function

$$\overset{*}{w}{}_k^{+}(z) = \frac{A_k}{2\pi}\ln\frac{a_k(z)}{(z-z_k)}, \quad \overset{*}{w}{}_k^{-} = \frac{A_k}{2\pi}\ln a_k(z)(z-\bar{z}_k) \qquad (\operatorname{Im} z_k > 0),$$

without violating the conditions (1.1), (1.3). The function $a_k(z) \neq 0$ is regular in E(C) and in the vicinity $z = \infty$ expands into the series

$$a_k(z) = z + a_{0k} + \frac{a_{1k}}{z} + \ldots$$

In the derivation of equations (2.4), (2.5) the presence of the function $\overset{*}{w}_k(z) \neq 0$ in the sum (2.1) leads to the appearance of additional terms in q_k and $\tilde{q}_k$:

$$q_k^*(x) = \frac{A_k}{\pi}\mathrm{Re}\left\{m_1 \ln\frac{x-\bar{z}_k}{a_k(x)} + e^{i\nu x}\int_{\infty}^{x}\left[2m_2\frac{a_k'(u)}{a_k(u)} + \frac{1-m_1}{u-\bar{z}_k}\right]e^{-i\nu u}\,du\right\},$$

$$\tilde{q}_k^*(x_1) = -2\mathrm{Re}\,[w_k^*(x_1)].$$

We note that in the derivation of equation (2.4) the term with $\ln(z - z_k)$ is not subjected to the conjugation operation.

3. Calculation Examples. We shall assume that the oscillating body is an elliptical cylinder $x^2/a^2 + (y + h)^2/b^2 = 1$, where a and b are, respectively, the semi-major and semi-minor axes of the ellipse, h is the distance of the center of the ellipse from the interface.

The function $F(z, \tau)$ for the elliptical cylinder has the form

$$F(z,t) = -\left\{\chi(z,\tau) - \frac{\bar{\zeta}'(\tau)}{\bar{\zeta}^2(\tau)[\zeta(z) - 1/\bar{\zeta}(\tau)]}\right\},$$
$$\chi(z,\tau) = \frac{\zeta(z) - \zeta(\tau) - \zeta'(\tau)(z-\tau)}{(z-\tau)[\zeta(z) - \zeta(\tau)]}; \qquad (3.1)$$

$$\zeta = \frac{z + ih + \sqrt{(z+ih)^2 - c^2}}{a+b}, \quad c^2 = a^2 - b^2. \qquad (3.2)$$

The function $\bar{F}(z, \tau_1)$ is written similarly to (3.1) with account for the replacement of τ by τ_1 — iH. Since there are no deformations, then $A_k = 0$, and it is not difficult to construct $\overset{0}{w}_k(z)$ [8], using the mapping (3.2).

The basic hydrodynamic characteristics of the oscillating profile include the added mass coefficients

$$\lambda_{ij} = -\rho^+ \int_C \frac{\partial\varphi_i^*}{\partial n}\varphi_j^*\,ds,$$

where $\overset{*}{\varphi}_i(x, y)$ ($i = \overline{1,3}$) are the potentials for motion along the x and y axes and for rotation, which are defined by the potentials $\varphi_1(x, y)$, $\varphi_2(x, y)$.

From the formula

$$\eta(x,t) = -\frac{1}{g(\rho^+ - \rho^-)}\left(\rho^-\frac{\partial\varphi^-(x,0,t)}{\partial t} - \rho^+\frac{\partial\varphi^+(x,0,t)}{\partial t}\right)$$

we can determine the shape of the fluid interface.

An algorithm and a program in the Fortran language were developed for the solution of the systems of integral equations and the calculation of the hydrodynamic characteristics. Figures 1-4 show examples of the calculations.

Figure 1 shows the calculated values of the nonzero added mass coefficients $\bar{\lambda}_{11} = \lambda_{11}/\pi\rho^+b^2$, $\bar{\lambda}_{22} = \bar{\lambda}_{11}$ (b = 1 is the radius of the circle) of a circular cylinder in a two-layer infinitely deep liquid (curve 1 is for $\bar{\rho} = \rho^-/\rho^+ = 0.97$, and curve 2 is for $\bar{\rho} = 0$) at the distance $\bar{h} = h/b = 2$ from the interface as a function of the quantity νb. We note that the calculations for the free surface agree well with the analytic results [3], indicated by the crosses. The calculations of the nonzero added mass coefficients of an elliptical cylinder a/b = 2 with h/b = 2, H = ∞ for $\bar{\rho} = 0.97$ and 0 are shown in Figs. 2 and 3, where $\bar{\lambda}_{ij} = \lambda_{ij}/\pi\rho^+b^n$. For $\bar{\lambda}_{11}$, $\bar{\lambda}_{22}$ n = 2, for $\bar{\lambda}_{31}$ n = 3, for $\bar{\lambda}_{33}$ n = 4.

In Fig. 4 the solid lines represent examples of the calculation of the interface of the fluids ($\bar{\rho} = 0.97$) with h/b = 3, H/b = 4.5, νb = 0.6 for the horizontal-vertical oscillations of an elliptical cylinder a/b = 2 in accordance with the law $x_0 = -h + \varepsilon \sin \omega t$ for $\bar{t} = t\omega = \pi(n - 1)/2$ ($n = \overline{1,4}$), the dashed curves are the result for an infinitely deep liquid. We note that the influence of the bottom shows up in increase of the amplitude of the waves on one side of the profile and reduction of their amplitude on the other side of the profile.

In conclusion we emphasize the promising nature of the method of modeling the boundaries using the singularities for the solution of the problems of ship hydrodynamics, in which the translational motion of the body and its oscillations are taken into account.

REFERENCES

1. N. E. Kochin, "The plane problem of steady oscillations of bodies beneath the free surface of an incompressible liquid," Collected Works, Vol. 2, Izd-vo Akad. Nauk SSSR, Moscow-Leningrad (1949).
2. M. D. Khaskind, "The plane problem of the oscillations of a body beneath the surface of a liquid of finite depth," PMM, **8**, No. 4 (1944).
3. R. Eatock Taylor and C. S. Hu, "Multipole expansions for wave diffraction and radiation in deep water," Ocean Eng., **18**, No. 3 (1991).
4. I. V. Sturova, "Influence of regular waves on a submerged body traveling in a stratified fluid," in: Computational Technologies [in Russian], **1**, No. 3, Novosibirsk (1992).
5. G. G. Tumashev, N. D. Cherepenin, and M. V. Lotfullin, "A method for solving the problems of the motion of a system of profiles near a fluid interface," in Mechanics of Continua: Proceed. of Repub. Scientific-Engineering Conference [in Russian], Naberezhnye Chelny (1982).
6. N. D. Cherepenin, "On the oscillations of a profile near the interface of two fluids," Proceed. of Seminar on Boundary Problems [in Russian], No. 11, Kazan (1974).
7. M. V. Lotfullin and S. I. Filippov, "Modeling of interfaces in the problems of ponderable fluid flow around a wing profile," PMTF, No. 4 (1992).
8. L. I. Sedov, Plane Problems of Hydrodynamics and Aerodynamics [in Russian], Gostekhteoretizdat (1950).